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Cálculo III - Ay. 01
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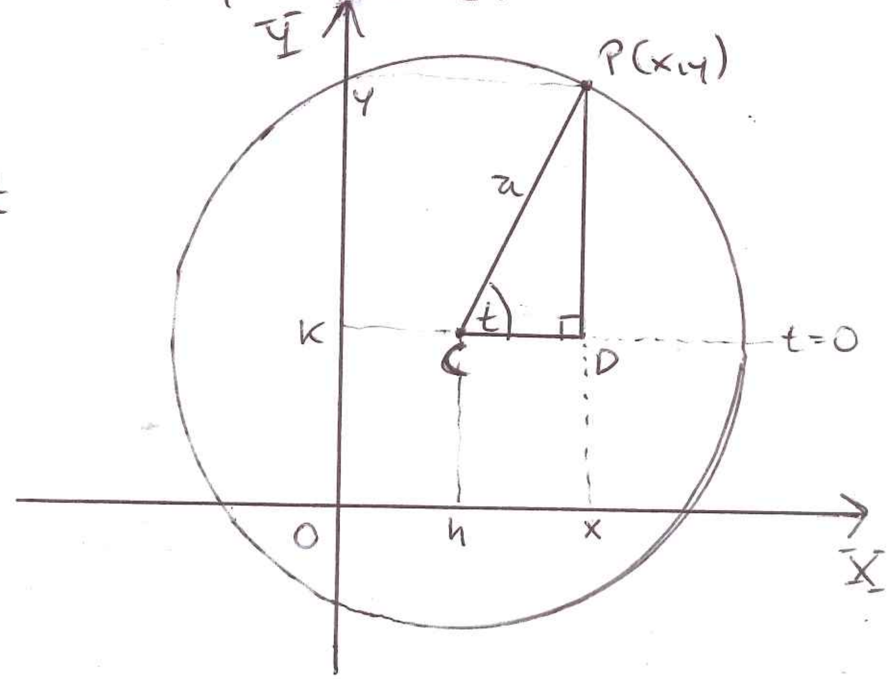
Ecuaciones Paramétricas

① Hallar las ecuaciones paramétricas de una circunferencia ζ, con centro (h,k) y radio a.



Sol: Para el triángulo CDP se tendrá:

$$\frac{x-h}{a} = \cos t \quad , \quad \frac{y-k}{a} = \operatorname{sen} t$$

$$\Rightarrow \begin{cases} x-h = a\cos t \\ y-k = a \operatorname{sen} t \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} x(t) = h + a\cos t \\ y(t) = k + a\operatorname{sen} t \end{cases}}$$

② Determinar $\frac{dy}{dx}$ para las siguientes curvas:

Ⓐ $\begin{cases} x(t) = a\cos t \\ y(t) = b \operatorname{sen} t \end{cases}$ (Elipse)

Ⓑ $\begin{cases} x(t) = a\sec t \\ y(t) = b\tan t \end{cases}$ (Hipérbola)

Ⓒ $\begin{cases} x(t) = \dfrac{t^2}{2p} \\ y(t) = t \end{cases}$ (Parábola)

Ⓓ $\begin{cases} x(t) = a\cos^3 t \\ y(t) = a\operatorname{sen}^3 t \end{cases}$ (Astroide)

Ⓔ $\begin{cases} x(t) = a\cos t + at\operatorname{sen} t \\ y(t) = a\operatorname{sen} t - at\cos t \end{cases}$ (Evolvente de la circunferencia)

Sol: (A) $\begin{cases} x(t) = a\cos t \\ y(t) = b\,\text{sen}\,t \end{cases}$

$$\frac{dx}{dt} = -a\,\text{sen}\,t \,, \quad \frac{dy}{dt} = b\cos t \implies \frac{dy}{dx} = \frac{b\cos t}{-a\,\text{sen}\,t} = -\frac{b}{a}\cot(t)$$

(B) $\begin{cases} x(t) = a\sec t \\ y(t) = b\tan t \end{cases}$

$$\frac{dx}{dt} = a\sec t\cdot\tan t \,, \quad \frac{dy}{dt} = b\cdot\sec^2 t \implies \frac{dy}{dx} = \frac{b\sec^{\cancel{2}} t}{a\,\cancel{\sec t}\tan t} = \frac{b}{\cos t}\cdot\frac{\cancel{\cos t}}{a\,\text{sen}\,t}$$

$$\boxed{\frac{dy}{dx} = \frac{b}{a}\csc t}$$

(C) $\begin{cases} x(t) = \dfrac{t^2}{2p} \\ y(t) = t \end{cases}$

$$\frac{dx}{dt} = \frac{\cancel{2}t}{\cancel{2}p} = \frac{t}{p} \,, \quad \frac{dy}{dt} = 1 \implies \frac{dy}{dx} = \frac{1}{\frac{t}{p}} = \frac{p}{t}$$

(D) $\begin{cases} x(t) = \\ y(t) = a\,\text{sen}^3 t \end{cases}$

$$\frac{dx}{dt} = 3a\cos^2 t\,(-\text{sen}\,t) = -3a\cos^2 t\,\text{sen}\,t$$

$$\frac{dy}{dt} = 3a\,\text{sen}^2 t\,(\cos t) = 3a\cos t\,\text{sen}^2 t$$

$$\implies \frac{dy}{dx} = \frac{\cancel{3a}\,\cancel{\cos t}\,\text{sen}^{\cancel{2}} t}{-\cancel{3a}\cos^{\cancel{2}} t\,\cancel{\text{sen}\,t}} = -\tan t$$

(E) $\begin{cases} x(t) = a\cos t + at\,\text{sen}\,t \\ y(t) = a\,\text{sen}\,t - at\cos t \end{cases}$

$$\frac{dx}{dt} = -a\,\text{sen}\,t + at\cos t$$

$$\frac{dy}{dt} = a\cos t + at\,\text{sen}\,t$$

$$\implies \frac{dy}{dx} = \frac{a\cos t + at\,\text{sen}\,t}{-(a\,\text{sen}\,t - at\cos t)} \left(= -\frac{x}{y}\right)$$

③ Determinar la longitud de arco pedida:

Ⓐ $\begin{cases} x(t) = h + a\cos t \\ y(t) = k + a\,\text{sen}\,t \end{cases}, \quad 0 \leq t \leq 2\pi$

Sol: $\frac{dx}{dt} = -a\,\text{sen}\,t \qquad \frac{dy}{dt} = a\cos t$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{(-a\,\text{sen}\,t)^2 + (\ \cos t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2\,\text{sen}^2 t + a^2\cos^2 t}\,dt$$

$$= \int_0^{2\pi} \sqrt{a^2(\text{sen}^2 t + \cos^2 t)}\,dt$$

$$= \int_0^{2\pi} a\,dt = at\Big|_0^{2\pi} = \boxed{2\pi a = L}$$

Ⓑ $\begin{cases} x(t) = 3t - t^3 \\ y(t) = 3t^2 \end{cases}, \quad 0 \leq t \leq 2$

Sol: $\frac{dx}{dt} = 3 - 3t^2 \qquad \frac{dy}{dt} = 6t$

$$\Rightarrow L = \int_0^2 \sqrt{(3-3t^2)^2 + (\ 6t\ )^2}\,dt$$

$$\int_0^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\,dt$$

$$= \int_0^2 \sqrt{9 + 18t^2 + 9t^4}\,dt$$

$$= \int_0^2 \sqrt{9(1 + 2t^2 + t^4)}\,dt$$

$$= \int_0^2 3\,(t^2+1)\,dt$$

$$\Rightarrow L = \cancel{3}\frac{t^3}{\cancel{3}} + 3t\Big|_0^2$$

$$= 2^3 + 3\cdot 2$$

$$L = 8 + 6$$

$$\boxed{L = 14}$$

(c) $\begin{cases} x(t) = e^t - t \\ y(t) = 4e^{t/2} \end{cases}$, $-8 \leq t \leq 3$.

Sol: $\frac{dx}{dt} = e^t - 1$ $\frac{dy}{dt} = \cancel{4}e^{t/2} \cdot \frac{1}{\cancel{2}} = 2e^{t/2}$

$$\Rightarrow L = \int_{-8}^{3} \sqrt{(e^t-1)^2 + (2e^{t/2})^2}\, dt$$

$$= \int_{-8}^{3} \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= \int_{-8}^{3} \sqrt{e^{2t} + 2e^t + 1}\, dt$$

$$= \int_{-8}^{3} \sqrt{(e^t+1)^2}\, dt$$

$$= \int_{-8}^{3} (e^t + 1)\, dt$$

$$= e^t + t \Big|_{-8}^{3}$$

$$= (e^3 + 3) - (e^{-8} + -8)$$

$$L = e^3 - \frac{1}{e^8} - 5$$